THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH2230A Complex Variables with Applications 2017-2018
Suggested Solution to Mid-term Examination

1. (a) Note that

$$\begin{aligned} & |z| = 3|z+i| \\ \iff & x^2+y^2 = 9(x^2+(y+1)^2) \\ \iff & x^2+\left(y+\frac{9}{8}\right)^2 = \frac{9}{64} \end{aligned}$$

Therefore, this equation represents a circle centred at $z = -\frac{9}{8}i$ with radius $\frac{3}{8}$. (The graph is omitted.)

(b) Since $\frac{\sqrt{2}i}{1+i} = e^{i\left(\frac{\pi}{4}+2k\pi\right)}$ for any $k \in \mathbb{Z}$, we have

$$\begin{aligned} \left(\frac{\sqrt{2}i}{1+i}\right)^{\frac{1}{6}} &= \left\{e^{i\frac{\pi}{24}}, e^{i\left(\frac{\pi}{24}+\frac{2\pi}{6}\right)}, e^{i\left(\frac{\pi}{24}+\frac{4\pi}{6}\right)}, e^{i\left(\frac{\pi}{24}+\frac{6\pi}{6}\right)}, e^{i\left(\frac{\pi}{24}+\frac{8\pi}{6}\right)}, e^{i\left(\frac{\pi}{24}+\frac{10\pi}{6}\right)}\right\} \\ &= \left\{e^{i\frac{\pi}{24}}, e^{i\frac{9\pi}{24}}, e^{i\frac{17\pi}{24}}, e^{i\frac{25\pi}{24}}, e^{i\frac{33\pi}{24}}, e^{i\frac{41\pi}{24}}\right\} \end{aligned}$$

2. Given that $f(z) = e^{x^2-y^2+a}[\cos(2xy+b) + i\sin(2xy+b)]$. We have

$$u(x,y) = e^{x^2-y^2+a}\cos(2xy+b) \text{ and } v(x,y) = e^{x^2-y^2+a}\sin(2xy+b).$$

Note that

$$\begin{aligned} u_x &= e^{x^2-y^2+a}[2x\cos(2xy+b) - 2y\sin(2xy+b)] = v_y \\ u_y &= e^{x^2-y^2+a}[-2y\cos(2xy+b) - 2x\sin(2xy+b)] = -v_x \end{aligned}$$

Since u_x, u_y, v_x and v_y are continuous and satisfy the Cauchy-Riemann equation for all $z \in \mathbb{C}$, $f(z)$ is an analytic function over $\mathbb{C}$.

Furthermore,

$$\begin{aligned} f'(z) &= u_x + iv_x \\ &= e^{x^2-y^2+a}[2x\cos(2xy+b) - 2y\sin(2xy+b)] + ie^{x^2-y^2+a}[2y\cos(2xy+b) + 2x\sin(2xy+b)] \\ &= e^{x^2-y^2+a}[\cos(2xy+b) + i\sin(2xy+b)](2x + i(2y)) \\ &= f(z)(2z) \end{aligned}$$

So we have $\frac{f'(z)}{f(z)} = 2z$.

3. For the function $\tanh z = \dfrac{\sinh z}{\cosh z}$,

$$\begin{aligned}
\{\text{zeros of } \tanh z\} &= \{\text{zeros of } \sinh z\} \\
&= \{\text{zeros of } -i\sin(iz)\} \\
&= \{z \mid iz = n\pi \text{ for some } n \in \mathbb{Z}\} \\
&= \{z \mid z = n\pi i \text{ for some } n \in \mathbb{Z}\}
\end{aligned}$$

$$\begin{aligned}
\{\text{singularity of } \tanh z\} &= \{\text{zeros of } \cosh z\} \\
&= \{\text{zeros of } \cos(iz)\} \\
&= \{z \mid -iz = \frac{\pi}{2} + n\pi \text{ for some } n \in \mathbb{Z}\} \\
&= \{z \mid z = \left(n + \frac{1}{2}\right)\pi i \text{ for some } n \in \mathbb{Z}\}
\end{aligned}$$

4. For the function $f(z) = \dfrac{\mathrm{Log}(1+z)}{z^2 - i}$, note that the function is not well-defined if

   (i) $z^2 - i = 0$;

   (ii) $\mathrm{Log}(1+z)$ is not well-defined.

   For (i), note that $z^2 = i = e^{i(\frac{\pi}{2})}$ implies $z = \pm e^{i(\frac{\pi}{4})} = \pm\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$.

   For (ii), $\mathrm{Log}(1+z)$ is not well-defined if and only if $(1+z) = -r$ for some $r \geq 0$, i.e. $z = -1-r$ for some $r \geq 0$.

   As a result, the maximum domain of $f(z)$ is given by

$$\mathbb{C}\backslash\left(\left\{\pm\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)\right\} \cup \{z = -1-r \mid r \geq 0\}\right).$$

   (The graph is omitted.)

5. For the contour C parametrized by $\gamma(\theta) = i + e^{i\theta}$, where $\theta \in [0, \frac{\pi}{2}]$,

$$\begin{aligned}
&\int_C (|z-i|^4 - \bar{z})dz \\
&= \int_0^{\frac{\pi}{2}} (|e^{i\theta}|^4 - (\overline{i + e^{i\theta}}))d(i + e^{i\theta}) \\
&= \int_0^{\frac{\pi}{2}} [1 - (\cos\theta - i(1+\sin\theta))]ie^{i\theta}d\theta \\
&= i\int_0^{\frac{\pi}{2}} [(1-\cos\theta) + i(1+\sin\theta)](\cos\theta + i\sin\theta)d\theta \\
&= i\int_0^{\frac{\pi}{2}} [[\cos\theta(1-\cos\theta) - \sin\theta(1+\sin\theta)] + i[\cos\theta(1+\sin\theta) + \sin\theta(1-\cos\theta)]]\, d\theta \\
&= -\int_0^{\frac{\pi}{2}} (\cos\theta + \sin\theta)d\theta + i\int_0^{\frac{\pi}{2}} (\cos\theta - \sin\theta - 1)d\theta \\
&= -2 - \frac{\pi}{2}i
\end{aligned}$$